

Section A: Pure Mathematics

- 1 A number of the form $1/N$, where N is an integer greater than 1, is called a *unit fraction*.

Noting that

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{6} \quad \text{and} \quad \frac{1}{3} = \frac{1}{4} + \frac{1}{12},$$

guess a general result of the form

$$\frac{1}{N} = \frac{1}{a} + \frac{1}{b} \quad (*)$$

and hence prove that any unit fraction can be expressed as the sum of two distinct unit fractions.

By writing (*) in the form

$$(a - N)(b - N) = N^2$$

and by considering the factors of N^2 , show that if N is prime, then there is only one way of expressing $1/N$ as the sum of two distinct unit fractions.

Prove similarly that any fraction of the form $2/N$, where N is prime number greater than 2, can be expressed uniquely as the sum of two distinct unit fractions.

- 2 Prove that if $(x - a)^2$ is a factor of the polynomial $p(x)$, then $p'(a) = 0$. Prove a corresponding result if $(x - a)^4$ is a factor of $p(x)$.

Given that the polynomial

$$x^6 + 4x^5 - 5x^4 - 40x^3 - 40x^2 + 32x + k$$

has a factor of the form $(x - a)^4$, find k .

- 3 The lengths of the sides BC , CA , AB of the triangle ABC are denoted by a , b , c , respectively. Given that

$$b = 8 + \epsilon_1, \quad c = 3 + \epsilon_2, \quad A = \pi/3 + \epsilon_3,$$

where ϵ_1 , ϵ_2 , and ϵ_3 are small, show that $a \approx 7 + \eta$, where $\eta = (13\epsilon_1 - 2\epsilon_2 + 24\sqrt{3}\epsilon_3)/14$.

Given now that

$$|\epsilon_1| \leq 2 \times 10^{-3}, \quad |\epsilon_2| \leq 4 \cdot 9 \times 10^{-2}, \quad |\epsilon_3| \leq \sqrt{3} \times 10^{-3},$$

find the range of possible values of η .

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4 Prove that

$$(\cos \theta + i \sin \theta)(\cos \phi + i \sin \phi) = \cos(\theta + \phi) + i \sin(\theta + \phi)$$

and that, for every positive integer n ,

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

By considering $(5 - i)^2(1 + i)$, or otherwise, prove that

$$\arctan(7/17) + 2 \arctan(1/5) = \pi/4.$$

Prove also that

$$3 \arctan(1/4) + \arctan(1/20) + \arctan(1/1985) = \pi/4.$$

[Note that $\arctan \theta$ is another notation for $\tan^{-1} \theta$.]

5 It is required to approximate a given function $f(x)$, over the interval $0 \leq x \leq 1$, by the linear function λx , where λ is chosen to minimise

$$\int_0^1 (f(x) - \lambda x)^2 dx.$$

Show that

$$\lambda = 3 \int_0^1 xf(x) dx.$$

The residual error, R , of this approximation process is such that

$$R^2 = \int_0^1 (f(x) - \lambda x)^2 dx.$$

Show that

$$R^2 = \int_0^1 (f(x))^2 dx - \frac{1}{3}\lambda^2.$$

Given now that $f(x) = \sin(\pi x/n)$, show that (i) for large n , $\lambda \approx \pi/n$ and (ii) $\lim_{n \rightarrow \infty} R = 0$.

Explain why, prior to any calculation, these results are to be expected.

[You may assume that, when θ is small, $\sin \theta \approx \theta - \theta^3/6$ and $\cos \theta \approx 1 - \theta^2/2$.]

6 Show that

$$\sin \theta = \frac{2t}{1+t^2}, \quad \cos \theta = \frac{1-t^2}{1+t^2}, \quad \frac{1+\cos \theta}{\sin \theta} = \tan(\pi/2 - \theta/2),$$

where $t = \tan(\theta/2)$.

Use the substitution $t = \tan(\theta/2)$ to show that, for $0 < \alpha < \pi/2$,

$$\int_0^{\pi/2} \frac{1}{1 + \cos \alpha \sin \theta} d\theta = \frac{\alpha}{\sin \alpha},$$

and deduce a similar result for

$$\int_0^{\pi/2} \frac{1}{1 + \sin \alpha \cos \theta} d\theta.$$

7 The line l has vector equation $\mathbf{r} = \lambda \mathbf{s}$, where

$$\mathbf{s} = (\cos \theta + \sqrt{3}) \mathbf{i} + (\sqrt{2} \sin \theta) \mathbf{j} + (\cos \theta - \sqrt{3}) \mathbf{k}$$

and λ is a scalar parameter. Find an expression for the angle between l and the line $\mathbf{r} = \mu(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$. Show that there is a line m through the origin such that, whatever the value of θ , the acute angle between l and m is $\pi/6$.

A plane has equation $x - z = 4\sqrt{3}$. The line l meets this plane at P . Show that, as θ varies, P describes a circle, with its centre on m . Find the radius of this circle.

8 (i) Let y be the solution of the differential equation

$$\frac{dy}{dx} + 4x e^{-x^2} (y+3)^{\frac{1}{2}} = 0 \quad (x \geq 0),$$

that satisfies the condition $y = 6$ when $x = 0$. Find y in terms of x and show that $y \rightarrow 1$ as $x \rightarrow \infty$.

(ii) Let y be any solution of the differential equation

$$\frac{dy}{dx} - x e^{6x^2} (y+3)^{1-k} = 0 \quad (x \geq 0).$$

Find a value of k such that, as $x \rightarrow \infty$, $e^{-3x^2} y$ tends to a finite non-zero limit, which you should determine.

Section B: Mechanics

- 9 In an aerobatics display, Jane and Karen jump from a great height and go through a period of free fall before opening their parachutes. While in free fall at speed v , Jane experiences air resistance kv per unit mass but Karen, who spread-eagles, experiences air resistance $kv + (2k^2/g)v^2$ per unit mass. Show that Jane's speed can never reach g/k . Obtain the corresponding result for Karen.

Jane opens her parachute when her speed is $g/(3k)$. Show that she has then been in free fall for time $k^{-1} \ln(3/2)$.

Karen also opens her parachute when her speed is $g/(3k)$. Find the time she has then been in free fall.

- 10 A long light inextensible string passes over a fixed smooth light pulley. A particle of mass 4 kg is attached to one end A of this string and the other end is attached to a second smooth light pulley. A long light inextensible string BC passes over the second pulley and has a particle of mass 2 kg attached at B and a particle of mass of 1 kg attached at C . The system is held in equilibrium in a vertical plane. The string BC is then released from rest. Find the accelerations of the two moving particles.

After T seconds, the end A is released so that all three particles are now moving in a vertical plane. Find the accelerations of A , B and C in this second phase of the motion. Find also, in terms of g and T , the speed of A when B has moved through a total distance of $0.6gT^2$ metres.

- 11 The string AP has a natural length of 1.5 metres and modulus of elasticity equal to $5g$ newtons. The end A is attached to the ceiling of a room of height 2.5 metres and a particle of mass 0.5 kg is attached to the end P . The end P is released from rest at a point 0.5 metres above the floor and vertically below A . Show that the string becomes slack, but that P does not reach the ceiling.

Show also that while the string is in tension, P executes simple harmonic motion, and that the time in seconds that elapses from the instant when P is released to the instant when P first returns to its original position is

$$\left(\frac{8}{3g}\right)^{\frac{1}{2}} + \left(\frac{3}{5g}\right)^{\frac{1}{2}} \left(\pi - \arccos(3/7)\right).$$

[Note that $\arccos x$ is another notation for $\cos^{-1} x$.]

Section C: Probability and Statistics

- 12 *Tabulated values of $\Phi(\cdot)$, the cumulative distribution function of a standard normal variable, should not be used in this question.*

Henry the commuter lives in Cambridge and his working day starts at his office in London at 0900. He catches the 0715 train to King's Cross with probability p , or the 0720 to Liverpool Street with probability $1 - p$. Measured in minutes, journey times for the first train are $N(55, 25)$ and for the second are $N(65, 16)$. Journey times from King's Cross and Liverpool Street to his office are $N(30, 144)$ and $N(25, 9)$, respectively. Show that Henry is more likely to be late for work if he catches the first train.

Henry makes M journeys, where M is large. Writing A for $1 - \Phi(20/13)$ and B for $1 - \Phi(2)$, find, in terms of A , B , M and p , the expected number, L , of times that Henry will be late and show that for all possible values of p ,

$$BM \leq L \leq AM.$$

Henry noted that in $3/5$ of the occasions when he was late, he had caught the King's Cross train. Obtain an estimate of p in terms of A and B .

[A random variable is said to be $N(\mu, \sigma^2)$ if it has a normal distribution with mean μ and variance σ^2 .]

- 13** A group of biologists attempts to estimate the magnitude, N , of an island population of voles (*Microtus agrestis*). Accordingly, the biologists capture a random sample of 200 voles, mark them and release them. A second random sample of 200 voles is then taken of which 11 are found to be marked. Show that the probability, p_N , of this occurrence is given by

$$p_N = k \frac{((N - 200)!)^2}{N!(N - 389)!},$$

where k is independent of N .

The biologists then estimate N by calculating the value of N for which p_N is a maximum. Find this estimate.

All unmarked voles in the second sample are marked and then the entire sample is released. Subsequently a third random sample of 200 voles is taken. Write down the probability that this sample contains exactly j marked voles, leaving your answer in terms of binomial coefficients.

Deduce that

$$\sum_{j=0}^{200} \binom{389}{j} \binom{3247}{200-j} = \binom{3636}{200}.$$

- 14** The random variables $X_1, X_2, \dots, X_{2n+1}$ are independently and uniformly distributed on the interval $0 \leq x \leq 1$. The random variable Y is defined to be the median of $X_1, X_2, \dots, X_{2n+1}$. Given that the probability density function of Y is $g(y)$, where

$$g(y) = \begin{cases} ky^n(1-y)^n & \text{if } 0 \leq y \leq 1 \\ 0 & \text{otherwise,} \end{cases}$$

use the result

$$\int_0^1 y^r(1-y)^s dy = \frac{r!s!}{(r+s+1)!}$$

to show that $k = (2n+1)!/(n!)^2$, and evaluate $E(Y)$ and $\text{Var}(Y)$. Hence show that, for any given positive number d , the inequality

$$P(|Y - 1/2| < d/\sqrt{n}) < P(|\bar{X} - 1/2| < d/\sqrt{n})$$

holds provided n is large enough, where \bar{X} is the mean of $X_1, X_2, \dots, X_{2n+1}$.

[You may assume that Y and \bar{X} are normally distributed for large n .]